



BY ROGER GOODMAN

THE PYTHAGOREAN COMMA, THE SPIRAL OF FIFTHS AND THE ADVENT OF EQUAL TEMPERAMENT

*Pythagoras: Greek philosopher, fifth century BC.
Comma: A pause or a space between things.*

WARNING: This article may contain methods of math instruction. You may find the following content to be mathematically and musically challenging but I think you'll find it worth the effort.

Let's start with a little mathematical story:

NATIVE AMERICAN MATH

Three Native American women are sitting and talking. The first is sitting on a deerskin and she says, "I have a son who weighs 130 pounds." The second one is sitting on a bearskin and says, "I have a son who weighs 170 pounds." The third one is sitting on a hippopotamus skin and she says, "Well, I myself weigh 300 pounds." The question then is, "What basic principle of mathematics does this scene represent?" And the answer is... wait for it... "The sons of the squaws of the two hides is equal to the squaw of the hippopotamus."

Fig. 1 - Pythagoras

This joke is a reference pun. In order for it to work it is assumed that the recipient is familiar with the reference. In this case the reference is to the Pythagorean Theorem that states that, "The sum of the squares of the two sides is equal to the square of the hypotenuse." The nice thing about using this reference is that almost everyone who took high school algebra will know this, and even those who don't think they like math will still be able to get a good chuckle. But did you know that the Greek philosopher Pythagoras (c565BC-c490BC) from his observations in music, mathematics and astronomy, generalized that everything could be expressed in terms of numbers and numerical ratios. Numbers are not only symbols of reality, but also substances of real things; hence, he claimed, "All is number."

PYTHAGORAS

Pythagoras (see **Figure 1**) was the first person to study the notes produced by plucked strings of various lengths. He discovered the series of harmonic overtones and found they could be expressed as the ratios of small whole numbers (see previous columns in the Archives at www.folkworks.org). The frequencies of the notes of a perfect fifth interval are in the ratio of 3 to 2 (also noted as 3:2, 3/2 or 1.5). This is called the Pythagorean fifth, and was considered throughout the Middle Ages to be the most harmonious interval outside the octave. When tuning an instrument it was important to make sure that the octaves were true and that the fifths were as close to true as possible.

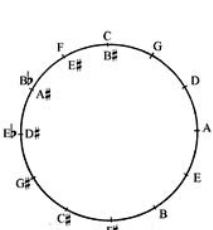


Fig. 2 The Circle of Fifths

When watching piano tuners you might notice that they first tune the instrument's central octave (the F-F octave around middle C) by going around the circle of fifths, a method of tuning that survives from the times of the ancient Greeks. But this is not as straightforward as you might think and here's why.

THE CIRCLE OF FIFTHS IS NOT REALLY A CIRCLE — IT IS MORE LIKE A SPIRAL.

It doesn't have closure. That is to say that the ends don't meet; instead, they overlap (see **Figures 2 & 3**). Twelve perfect fifths should add up to exactly seven octaves from where you begin — but it doesn't.

This is how it should work:

1	2	3	4	5	6	7	8	9	10	11	12	Perfect Fifths
C	G	D	A	E	B	F#	C#	G#	D#	A#	E#	B#
					Gb	Db	Ab	Eb	Bb	F	C	

1	2	3	4	5	6	7	Octaves
C	C	C	C	C	C	C	C

TABLE #1

But here's what really happens: start at the lowest frequency for C (32.7 Hz) on the piano and multiply by 1.5 (the ratio of 3/2) to get the fifth above that—the G (49.05 Hz)—then multiply that by 1.5 again to get to the next fifth up and so on. This is the result:

1	2	3	4	5	6	7	8	9	10	11	12	Perfect Fifths
C	G	D	A	E	B	F#	C#	G#	D#	A#	E#	B#
					Gb	Db	Ab	Eb	Bb	F	C	
32.7	73.5	165.5	372.4	838.0	1885.6	4742.7 Hz						
	49.0	110.3	248.3	558.7	1257.0	2828.4						

TABLE #2

In a similar fashion you get the octaves when you multiply by 2 (ratio 2/1) like this:

1	2	3	4	5	6	7	Octaves
C	C	C	C	C	C	C	C
32.7	65.4	130.8	261.6	523.2	1046.4	2092.8	4185.6 Hz

TABLE #3

Counting up by seven octaves (ratios of 2/1) from C 32.7 Hz winds up at C 4185.6 Hz but counting up by twelve fifths (ratios of 3/2) yields C 4242.7 Hz. This discrepancy is known as the Pythagorean Comma and has been a powerful challenge for instrument makers and tuners. Fixed note instruments like keyboards and fretted instruments must have a way to manage this error. The various schemes to do this are called temperaments. The Pythagoreans left the entire comma in the least used fifth between G# and Eb. This left all of the fifths in tune except for the G#-Eb interval which was extra large, rather dissonant and, some say, howled like a wolf. This "wolf fifth" was placed where it could most likely be avoided.

The necessity for temperaments arose because of the conflict between the desire to have certain intervals (especially the perfect fifth, the perfect fourth and the major thirds) be as harmonious as possible while still allowing one to play the instrument in as many different keys (i.e. starting the scale on as many different notes) as possible. Many different tempering schemes have come and gone over the past centuries of musical evolution. What they all have in common is the re-tuning of the scale so that some or all of the intervals are slightly "out-of-tune" but no particular ones are left grossly out-of-tune.

EQUAL TEMPERAMENT

Equal temperament is a method of tuning that spreads the Pythagorean comma equally over all twelve of the fifths, each one being flattened by 1/12 comma. This gives the equal tempered scale flexibility and simplicity at the expense of musical purity. Now twelve tempered fifths precisely equal seven octaves, which closes the circle of fifths. All intervals except the octaves are slightly, but acceptably, out of tune.

In modern music theory B# and C are considered to be enharmonic, that is notes that are named differently but sound the same. When scales were not tempered notes like B# and C were separate and distinct notes unto themselves. With equal temperament that distinction disappeared. Some of the very old keyboard instruments had two or three separate manuals (keyboards) and as many separate sets of strings to allow the instrument to play in more than one key. Contemporary with those instruments were instruments that had the "black keys" split into two notes to allow for the difference in the enharmonic notes like D# and Eb.

We no longer have to worry about tunings when changing keys or about the differences between enharmonic notes, but we may have lost some of the emotional impact that went with the old tunings. Composers back then believed that each key evoked a particular mood and would choose a key for the mood they wished to convey. I happen to think the trade off is worth it for the simplicity. Until next time please keep a good temperament and, of course, stay tuned.

PS: Try this on the Web: Google this fraction "531441/262144."

Roger Goodman is a musician, mathematician, punster, reader of esoteric books and sometime writer, none of which pays the mortgage. For that, he is a computer network guy for a law firm. He has been part of the Los Angeles old-time & contra-dance music community for over thirty years. While not a dancer, he does play fiddle, guitar, harmonica, mandolin, banjo & spoons. Roger has a penchant for trivia and obscure and sometimes tries to explain how the clock works when asked only for the time. He lives with his wife, Monika White, in Santa Monica.